Toppersnotes
Unleash the topper in you

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INNOVATIONS

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Vector space -

Let V be any non empty set and let $(F, +, \cdot)$ be any Field. Let $*$ and '$\circ$' be any two operation define

$$* : V \times V \longrightarrow V$$
$$\circ : F \times V \longrightarrow V$$

then V is said to be vector space over the Field F. if the following condition are satisfied.

(i) $(V, *)$ is an abelian group.

(ii) $(\alpha + \beta) \circ v = (\alpha \circ v) * (\beta \circ v)$

(iii) $\alpha \circ (u * v) = (\alpha \circ u) * (\alpha \circ v)$

(iv) $(\alpha \cdot \beta) \circ v = \alpha \circ (\beta \circ v)$

(v) $1 \cdot \circ u = u$

$\forall \alpha, \beta \in F$
$u, v \in V$
$1 =$ unity of the Field F.

Example :- $V = \mathbb{R}^+$, $F = (\mathbb{R}, +, \cdot)$

Define $* : V \times V \longrightarrow V$

$$u * v = u \cdot v$$

$$\circ : F \times V \longrightarrow V$$

$$\alpha \circ u = u^{\alpha}$$

(i) $(V, *)$ is an abelian group.

(ii) $(\alpha + \beta) \circ u = u^{\alpha+\beta} = u^{\alpha} \cdot u^{\beta} = (\alpha \circ u) * (\beta \circ u)$

(iii) $\alpha \circ (u * v) = (uv)^{\alpha} = u^{\alpha} \cdot v^{\alpha} = (\alpha \circ u) * (\alpha \circ v)$

(iv) $(\alpha \cdot \beta) \circ u = u^{\alpha\beta} = (u^{\beta})^{\alpha} = (\beta \circ u)^{\alpha} = \alpha \circ (\beta \circ u)$

(v) $1 \circ u = u' = u$.

V forms a vector space over the field F.

<u>Example:-</u> $V = \mathbb{R}^n$ $F = (\mathbb{R}, +, \cdot)$

Define $* : V \times V \longrightarrow V$

$(u_1, u_2 \dots u_n) * (v_1, v_2, \dots v_n)$

$= (u_1 + v_1, u_2 + v_2 \dots u_n + v_n)$

$\circ : F \times V \longrightarrow V$.

$\alpha \circ (u_1, u_2 \dots u_n) = (\alpha u_1, \alpha u_2, \dots \alpha u_n)$

1. $(V, *)$ forms abelian group.

(ii) $(\alpha + \beta) \circ u = (\alpha + \beta) \circ (u_1, u_2 \dots u_n)$

$= ((\alpha + \beta) u_1, (\alpha + \beta) u_2, \dots (\alpha + \beta) u_n)$

$= (\alpha u_1 + \beta u_1, \alpha u_2 + \beta u_2, \dots \alpha u_n + \beta u_n)$

$= (\alpha u_1, \alpha u_2 \dots \alpha u_n) + (\beta u_1, \beta u_2 \dots \beta u_n)$

$= \alpha \circ u_1 * \beta u_2$

iii. $\alpha \circ (u * v) = \alpha \circ (u_1 + v_1, u_2 + v_2, \dots u_n + v_n)$

$= (\alpha u_1 + \alpha v_1, \alpha u_2 + \alpha v_2, \dots \alpha u_n + \alpha v_n)$

$= (\alpha u_1, \alpha u_2 \dots \alpha u_n) + (\alpha v_1, \alpha v_2 \dots \alpha v_n)$

$= (\alpha \circ u) * (\alpha \circ v)$

iv. $(\alpha \cdot \beta) \circ u = \alpha \beta \circ (u_1, u_2 \dots u_n)$

$= (\alpha \beta u_1, \alpha \beta u_2 \dots \alpha \beta u_n)$

$= \alpha(\beta u_1 \quad \beta u_2, \ \dots \ \beta u_n)$
$(\beta o u_1 \quad \beta o u_2 \ \dots \ \beta o u_n)$

$= \alpha(\beta o u_1)$

$= \alpha o(\beta o u_1)$

v. $1 \cdot o u = (1 \cdot u_1 \quad 1 \cdot u_2 \dots \ 1 \cdot u)$

$= (u_1, u_2 \dots \ u_n)$

$1 \cdot u = u$

Hence V forms a vector space over the field F

<u>Example:-</u> $V = \mathbb{R}^n$, $\quad F = (\mathbb{C}, +, \cdot)$

Define $*: V \times V \longrightarrow V$

$(u_1, u_2 \dots u_n) * (v_1, v_2 \dots v_n) = (u_1+v_1, \dots u_n+v_n)$

$o: F \times V \longrightarrow V$

$\alpha \, o \, (u_1, u_2 \dots u_n) = (\alpha u_1, \alpha u_2 \dots \alpha u_n)$

Take $\alpha = i \in \mathbb{C}$, $\quad u = (1, 1, \dots 1) \in V$.

$i \, o \, (u_1, u_2 \dots u_n) = i \, o \, (1, 1, \dots 1)$

$= (i, i, i \dots i) \notin V$

Hence $(V, *, o)$ is not vector space.

<u>V(F)</u>	$\mathbb{Q} \subsetneq \mathbb{R} \subsetneq \mathbb{C}$
$\mathbb{Q}^n(\mathbb{R})$	✗
$\mathbb{R}^n(\mathbb{Q})$	✓
$\mathbb{C}^n(\mathbb{Q})$	✓
$\mathbb{C}^n(\mathbb{R})$	✓
$\mathbb{Q}^n(\mathbb{R})$	✗.

Example :- $V = \mathbb{R}$, $F = (\mathbb{R}, +, \cdot)$

Define $* : V \times V \longrightarrow V$

$u * v = u + v + 1$

$o : F \times V \longrightarrow V$

$\alpha o u = \alpha u + \alpha - 1$

Soln :

1. $(V, *)$ forms abelian group identity is -1 and inverse $-2-u$.

2. $(\alpha+\beta) o u = (\alpha+\beta) u + \alpha + \beta - 1$

$= \alpha u + \beta u + \alpha + \beta - 1$

$(\alpha o u) * (\beta o u) = (\alpha u + \alpha - 1) * (\beta u + \beta - 1)$

$= \alpha u + \alpha - 1 + \beta u + \beta - 1 + 1$

$= \alpha(u+\beta) + \alpha + \beta - 1$

3. $\alpha o (u * v) = \alpha o (u + v + 1)$

$= \alpha(u + v + 1) + \alpha - 1$

$= \alpha u + \alpha v + 2\alpha - 1$

$(\alpha o u) * (\alpha o v) = (\alpha u + \alpha + 1) * (\alpha v + \alpha + 1)$

$= \alpha u + \alpha - 1 + \alpha v + \alpha - 1 + 1$

$= \alpha(u + v) + 2\alpha - 1$

$\Rightarrow \alpha o (u * v) = (\alpha o u) * (\alpha o v)$

4. $\alpha\beta o u = \alpha\beta u + \alpha\beta - 1$

$\alpha o (\beta o u) = \alpha o (\beta u + \beta - 1)$

$= \alpha(\beta u + \beta - 1) + \alpha - 1$

$= \alpha\beta u + \alpha\beta - 1$

$\alpha\beta \circ u = \alpha \circ (\beta \circ u)$

5. $1 \circ u = u + 1 - 1 = u.$

$\therefore$ $(V, *, \circ)$ forms a vector space over the Field F.

<u>Example</u>: $V = \mathbb{R}$, $F = (\mathbb{R}, +, \cdot)$

Define $* : V \times V \longrightarrow V$

$u * v = u + v.$

$\circ : F \times V \longrightarrow V$

$\alpha \circ u = |\alpha| \cdot u.$

<u>Sol^n</u>:

1. $(V, *)$ forms an abelian group

2. $(\alpha + \beta) \circ u = |\alpha + \beta|\, u.$

$(\alpha \circ u) * (\beta \circ u) = |\alpha| u \circ |\beta| \cdot u$

$= |\alpha| u + |\beta| u$

$= (|\alpha| + |\beta|) \cdot u$

$|\alpha + \beta| \cdot u \neq (|\alpha| + |\beta|) \cdot u$

i.e. $\alpha = 1, \quad \beta = -1.$

$0 \cdot u \neq 2 \cdot u.$

Hence $(V, *, \circ)$ does not forms a vector space over $\mathbb{R}$.

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Example:- $V = \mathbb{R}^2$, $F = (\mathbb{R}, +, \cdot)$

Define $* : V \times V \longrightarrow V$

$(u_1, u_2) * (v_1, v_2) = (u_1 + v_1 \quad u_2 + v_2)$

$o : F \times V \longrightarrow V$

$\alpha \, o (u_1, u_2) = (\alpha u_1, 0)$

solution:-

1. $(V, *, o)$ forms an abelian group

2. $(\alpha + \beta) \, o \, (u) = (\alpha + \beta) \, o \, (u_1, u_2)$

$= (\alpha + \beta u_1, 0)$

$= (\alpha u_1 + \beta u_1, 0)$

$= (\alpha u_1, 0) + (\beta u_1, 0)$

$= \alpha \, o (u_1, u_2) + \beta o (u_1, u_2)$

$= \alpha \, o (u_1, u_2) * \beta \, o (u_1, u_2)$

$= (\alpha \, o \, u) * (\beta \, o \, u)$

3. $\alpha \, o (u * v) = \alpha \, o \left((u_1, u_2) * (v_1, v_2) \right)$

$= \alpha \, o \left(u_1 + v_1 \quad u_2 + v_2 \right)$

$= \left(\alpha (u_1 + v_1), 0 \right)$

$= \left(\alpha u_1 + \alpha v_1, 0 \right)$

$= (\alpha u_1, 0) + (\alpha v_1, 0)$

$= \alpha \, o (u_1, u_2) + \alpha \, o (v_1, v_2)$

$= (\alpha \, o \, u) * (\alpha \, o \, v)$

4 $(\alpha\cdot\beta)\circ u = (\alpha\cdot\beta)\circ(u_1,u_2) = (\alpha\beta u_1, 0)$

$(\alpha\circ u) * (\beta\circ u) = (\alpha\circ(u_1,u_2)) * \beta\circ(u_1,u_2)$

$= (\alpha u_1, 0) * (\beta u_1, 0)$

$= (\alpha u_1 + \beta u_1, 0)$

$= ((\alpha+\beta)u_1, 0)$

(5) $1\circ u = 1\circ(u_1,u_2) = (u_1,0)$

$1\circ u \neq u$

$\Rightarrow$ $(V,*,\circ)$ does not form vector space over the field F.

Example :- $V = C[0,1]$, $F = (\mathbb{R},+,\cdot)$

Define $* : V\times V \longrightarrow V$

$(f*g)(x) = f(x)+g(x)$

$\circ : F\times V \longrightarrow V$

$(\alpha\circ f)(x) = \alpha\cdot f(x)$.

Solution:-

1. $(V,*)$ forms an abelian group.

2. $(\alpha+\beta)\circ(f)(x) = (\alpha+\beta)f(x)$.

$= \alpha\cdot f(x) + \beta f(x)$

$= (\alpha\circ f)(x) + (\beta\circ f)(x)$

$= (\alpha\circ f) * (\beta\circ f)(x)$

$(\alpha+\beta)\circ f = (\alpha\circ f) * (\beta\circ f)$

3. $\alpha \circ (f * g)(x) = \alpha \circ (f(x) + g(x))$

$= \alpha \circ (f(x) + g(x))$

$= \alpha (f(x) + g(x))$

$= \alpha \cdot f(x) + \alpha \cdot g(x)$

$= \alpha \circ f(x) + \alpha \circ g(x)$

$= (\alpha \circ f)(x) * (\alpha \circ g)(x)$

$\alpha \circ (f * g)(x) = [(\alpha \circ f * (\alpha \circ g)](x)$

4. $((\alpha \cdot \beta) \circ f)(x) = \alpha \cdot \beta\, f(x)$

$= \alpha \cdot \beta f(x)$

$= \alpha \cdot (\beta \circ f) \cdot x.$

$= (\alpha \circ (\beta \circ f))x.$

5. $(1 \circ f)(x) = 1 \cdot f(x)$

$= f(x)$

Hence $(V, *, \circ)$ Forms a vector space over the Field. F.

Example:- $V = C[0,1]$, $F = (\mathbb{C}, +, \cdot)$

Define $*: V \times V \longrightarrow V$

$(f * g)(x) = f(x) + g(x)$

$\circ : F \times V \longrightarrow V$

$(\alpha \circ f)(x) = \alpha f(x).$

$(V, *)$ forms an abelian group.

Take $\alpha = i \in \mathbb{C}$

$f(x) = 2, \in V$

$(\alpha \circ f)(x) = \alpha \cdot f(x)$

$= i \cdot 2$

$= 2i \notin V$

$\Rightarrow$ Hence $(V, *, \circ)$ does not form vector space over the Field $\mathbb{C}$.

Example :- Let F be a Field of characteristic p (prime).

Let V(F) be a vector space.

Define

$* : V \times V \longrightarrow V$

$u * v =$ same as that of V

$\circ : F \times V \longrightarrow V.$

$\alpha \circ u = \alpha^{p} \cdot u.$

varify V(F) vector space or not.

Solution :-

(i) $(V, *)$ Forms an abelian group

(ii) $(\alpha+\beta) \circ u = (\alpha+\beta)^{p} \cdot u \cdot = (\alpha^{p} + \beta^{p}) u = \alpha^{p} \cdot u + \beta^{p} \cdot u$
$= (\alpha \circ u) * (\beta \circ u)$

$(\alpha \circ u) * (\beta \circ u) = \alpha^{p} u * \beta^{p} \cdot u$

$(\alpha \circ u) * (\beta \circ u) = (\alpha+\beta) \circ u.$

(iii) $\alpha \circ (u * v) = \alpha \circ (u + v) = \alpha^{p} (u + v)$

$= \alpha^{p} u + \alpha^{p} \cdot v.$

$= (\alpha \circ u) * (\alpha \circ v)$

(iv) $(\alpha \cdot \beta) \circ u = (\alpha\beta)^p \cdot u$

$= \alpha^p \cdot \beta^p \cdot u$

$= \alpha^p (\beta \circ u)$

$= \alpha \circ (\beta \circ u)$.

(v) $1 \cdot \circ u = 1^p \cdot u = u$

Hence V is vector space over F.

<u>Example</u> :- Let $a \in \mathbb{R}$, Let $T_a : \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$T_a(x) = a + x$.

$V = \{T_a : a \in \mathbb{R}\}$. and $F = (\mathbb{R}, +, \cdot)$

$* : V \times V \longrightarrow V$

$T_a * T_b = T_a(T_b)$.

$\circ : F \times V \longrightarrow V$

$\alpha \circ T_a = T_{\alpha a}$.

verify V is not vector space or vector.

1. $(V, *)$ forms an abelian group

2. $(\alpha + \beta) \circ T_a(x) = T_{(\alpha+\beta)a}(x)$

$= T_{\alpha a + a\beta}(x)$

$= \alpha a + a\beta + x$.

$(\alpha \circ T_a) * (\beta \circ T_a)(x) = (T_{\alpha a}) * (T_{\beta a})\, x$.

$= T_{\alpha a}(\beta a + x)$

$= \alpha a + \beta a + x$.

$\Rightarrow (\alpha + \beta) \circ T_a = (\alpha \circ T_a) * (\beta \circ T_a)$

3. $\alpha \circ (T_a * T_b)(x) = \alpha \circ (T_a(T_b))(x)$

$= \alpha \circ (T_a(b+x))$

$= \alpha \circ (a+b+x)$

$= \alpha \circ (a+b+x)$

$= \alpha \circ (T_{a+b}(x))$

$= T_{\alpha(a+b)}(x) = \alpha a + \alpha b + x.$

$(\alpha \circ T_a) * (\alpha \circ T_b)(x) = T_{\alpha a} * T_{\alpha b}(x)$

$= T_{\alpha a}(T_{\alpha b}(x))$

$= T_{\alpha a}(\alpha b + x)$

$= \alpha a + \alpha b + x.$

$\Rightarrow \alpha \circ (T_a * T_b) = (\alpha \circ T_a) * (\alpha \circ T_a)$

4. $(\alpha \cdot \beta) \circ T_a(x) = T_{\alpha\beta a}(x) = \alpha\beta a + x.$

$\alpha \circ (\beta \circ T_a(x)) = \alpha \circ (T_{\beta a}(x)) = T_{\alpha\beta a}(x) = \alpha\beta a + x$

(5) $1 \circ T_a = T_{1a} = T_a$

$\Rightarrow (V, *, \circ)$ forms a vector space over the field F.

<u>Note</u> :- Every field forms a vector space over its subfield.

<u>Example</u>:- $\mathbb{R}(\mathbb{Q})$, ✓

$\mathbb{Q}(\mathbb{R})$, ✗. $\mathbb{Q} \subseteq \mathbb{R}$

$\mathbb{Q}(\mathbb{C})$ ✗

$\mathbb{C}(\mathbb{R})$ ✓

$\mathbb{R}(\mathbb{C})$ ✗

$\mathbb{Q}(\sqrt{2})(\mathbb{Q})$ ✓

$\mathbb{Q}[i](\mathbb{Q})$ ✓

Sub space :- Let $V(F)$ be a vector space and let W be non-empty subset of V then W is said to be subspace of V if W itself forms a vector space over the field F.

Theorem :- Let $V(F)$ be a vector space and let W be any non empty subset of V then W is subspace of V iff $\alpha x+\beta y \in W$, $\forall \alpha, \beta \in F$ and $x, y \in W$.

conversely

Let $\alpha x+\beta y \in W$, $\forall \alpha, \beta \in F$ & $x, y \in W$.

Take $\alpha = 1$, $\beta = -1$

$x - y \in W$, $\forall x, y \in W$

$\Rightarrow$ W is subgroup of V

$\Rightarrow$ W is abelian subgroup of V.

Take $\beta = 0$

$\alpha \cdot x \in W$, $\forall \alpha \in F$, & $x \in W$.

$\Rightarrow$ W is closed under scalar multiplication.

Remaining all axioms also hold since elements of W are form V.

$\therefore$ W is subspace of V.

Above Result can be viewed as

W is subspace of V $\Leftrightarrow$ (i) $x+y \in W$, (ii) $\alpha x \in W$ $\Big\}$ $\forall \alpha, \beta \in F$ and $x, y \in V$.

Example:- $V = C([0,1])$, $F = (\mathbb{R}, +, \cdot)$

(1) $W_1 = \{ f \in V : f(1/2) \in \mathbb{Q} \}$. ✗

2. $W_2 = \{ f \in V : f(1/2) = 0 \}$ ✓

3. $W_3 = \{ f \in V : f(1/2) = 1 \}$. ✗

4. $W_4 = \{ f \in V : \int_0^1 f(t)\,dt = 1 \}$. ✗

5. $W_5 = \{ f \in V : \frac{df}{dt} = 0 \}$. ✓

Solution:-

1. $0 \in W_1$, $f(1/2) = 0 \in W_1$

$f(1/2) \in \mathbb{Q}$, $\alpha = \sqrt{2}$.

$\Rightarrow \alpha \cdot f(1/2) \notin \mathbb{Q}$.

$\Rightarrow$ W_1 is not subspace.

2. $f \in W_1$, $g \in W_1$ $\Rightarrow f(1/2) + g(1/2) = 0 + 0 = 0 \in W_1$

$\Rightarrow$ W_1 is subgroup of V.

$\alpha \in F$, $f \in W_1$

$\Rightarrow \alpha \cdot f(1/2) = \alpha \cdot 0 = 0 \in W_1$

$\Rightarrow$ Hence W_2 is subspace of V.

5. $f' \in W_5$ $g' \in W_5$ $\Rightarrow f' + g' = 0 + 0 = 0 \in W_5$

$\Rightarrow$ W_5 is subgroup of V.

$\alpha \in F$, $f' \in W_5$

$\Rightarrow \alpha \cdot F' = 0 \in W_5$, $\forall \alpha \in F$, $f' \in W_5$.

Hence W_5 is subspace of V.

Example: $V = M_2(\mathbb{R})$ $\quad F = (\mathbb{R}, +, \cdot)$

Define $*: V \times V \longrightarrow V$

$$\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} * \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} = \begin{bmatrix} a_1+a_2 & b_1+b_2 \\ c_1+c_2 & d_1+d_2 \end{bmatrix}$$

$\circ : F \times V \longrightarrow V$

$$\alpha \circ \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \alpha a & \alpha b \\ \alpha c & \alpha d \end{bmatrix}$$

clearly, $V(F)$ is a v.s.

1. $W_1 = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in V : a = d \right\}$ ✓

2. $W_2 = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in V : a = 0, b + c = 0 \right\}$ ✓

3. $W_3 = \{ A \in V : A^T = A \}$ ✓

4. $W_4 = \{ A \in V : A^T = -A \}$ ✓

5. $W_5 = \{ A \in V : \text{rank} A = n$, where n is fixed natural No. $\}$ ✗

6. $W_6 = \{ A \in V : \text{trace } A = 0 \}$ ✓ $\quad \text{Trace}(\alpha A + \beta B) = \alpha \text{ Trace} A + \beta \text{ Trace} B$

7. $W_7 = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a + b + c + d = 1 \right\}$ ✗

8. $W_8 = \{ A \in V : \det(A) = 0 \}$ ✗

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = 0 \Rightarrow |A| = 0$$

$$B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = 0 \Rightarrow |B| = 0$$

$$A + B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$W = W_3 \cap W_4$

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in W_3 \cap W_4$

$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in W_3, \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in W_4$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^T = \begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix}$$

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix}$$

$a = -a \Rightarrow 2a = 0 \Rightarrow a = 0$

$d = 0$

$b = -c$

$$A = \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix}$$

$A^T = A.$

$$\begin{bmatrix} 0 & -b \\ b & 0 \end{bmatrix} = \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}$$

$-b = b$

$2b = 0 \Rightarrow b = 0$

$\therefore \quad A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$W = W_3 \cap W_4 = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}$

forms a vector subspace.

Note: If A is skew symmetric

i.e. $A^T = -A$

$\Rightarrow a_{ij} = -a_{ji}$

In particular $j = i$

$a_{ii} = -a_{ii}$

$2a_{ii} = 0 \Rightarrow a_{ii} = 0$