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CSIR-NET

Council of Scientific & Industrial Research

MATHEMATICAL SCIENCE

VOLUME - V

SIERRA INNOVATIONS

INDEX

<u>Differential equation:</u>-
An equation between dependent variable and independent variable and derivative of dependent variable with r.t independent variable is called Differential equation.

<u>O.D.E</u> $y = y(x)$

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0$$

$$\frac{dy}{dx} + y = \sin x.$$

$y_1 = y_1(x)$

$y_2 = y_2(x)$

$$\frac{dy_1}{dx} + \frac{dy_2}{dx} = \cos x$$

$$\frac{d^2y_1}{dx^2} + \frac{d^2y_2}{dx^2} = 0$$

$z' = z(x,t)$

$$\frac{\partial z}{\partial x} + \frac{\partial^2 z}{\partial x^2} = 0$$

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial t} = 0$$

$z_1 = z_1(x,y)$

$z_2 = z_2(x,y)$

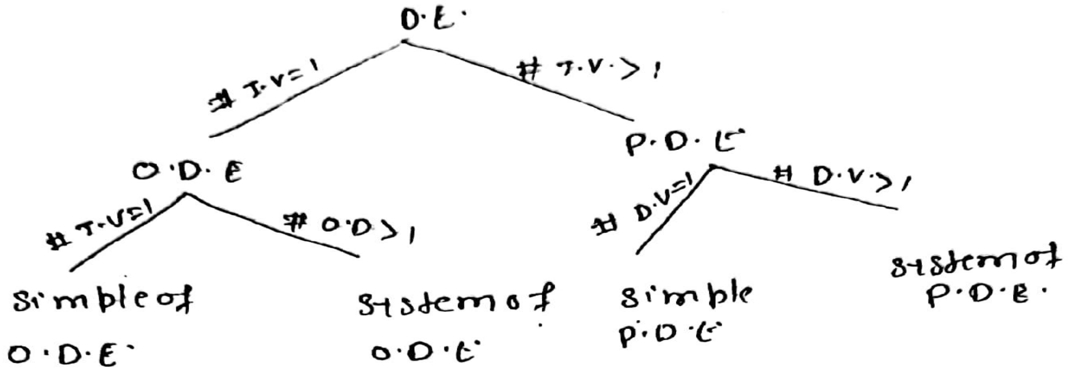
$$\frac{\partial z_1}{\partial x} + \frac{\partial z_2}{\partial t} = e^x$$

$$\frac{\partial^2 z_1}{\partial t^2} + e^x \frac{\partial z_2}{\partial x} = 0$$

<u>ordinary differential Equation.</u>
Any differential equation in which unique independent variable and total derivative dependent variable w.r.t. independent variable is called ordinary differential equation.

<u>Partial Differential Equation:</u>-
Any differential equation contain Partial derivative is called Partial differential equation.

classification of Diff. Equation.



formation of arbitary Diff. Equation.

Let $\psi(x, y, c_1, c_2 \cdots c_n) = 0$ be the relation —(1) between x, y and $c_1, c_2 - - - c_n$

where x – independent variable

y – dependent variable.

c_i's are arbitary constant.

Differentiate w.r.t to x.

$$\left.\begin{aligned} &\phi_1(x, y, y', c_2 \cdots c_n) = 0 \\ &\phi_2(x, y, y'', y', c_3 \cdots c_n) = 0 \\ &\vdots \\ &\phi_n(x, y, y', \cdots y^n, c_1 \cdots c_n) = 0 \end{aligned}\right\}$$

We eliminate $c_1 c_2, \cdots c_n$ from above $(n+1)$ equation.

We get

$$\boxed{\phi(x, y, y', \cdots y^n) = 0}$$

Example:- Find the differential E. from the relation

$ax^2 + by^2 = 1$

$2ax + b2y\cdot\frac{dy}{dx} = 0$

$2a + b2y\frac{d^2y}{dx^2} + 2b\left(\frac{dy}{dx}\right)^2 = 0$

$a + by\frac{d^2y}{dx^2} + b\cdot\left(\frac{dy}{dx}\right)^2 = 0$

$by\cdot\frac{dy}{dx} + by\frac{d^2y}{dx^2} + b\left(\frac{dy}{dx}\right)^2 = 0$

$b\left(\frac{y}{x}\frac{dy}{dx} + y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2\right] = 0$

$b \neq 0.$

$\frac{y}{x}\frac{dy}{dx} + y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0.$

$\boxed{yy' + xyy'' + xy'^2 = 0}$

Example:- Find the D.E. of the family of all circle in the xy-plane.

$ax^2+by^2+2hxy+2gx+2fy+c=0$

$\Delta = 0$

$\Delta \neq 0$

The family of circle is

$(x-a)^2 + (y-b)^2 = c^2$

$2(\cancel{x-a}) +$

$x^2+y^2+2gx+2fy+C=0$

$2x + 2y\frac{dy}{dx} + 2g + 2f\frac{dy}{dx} = 0$

$2 + 2y\frac{d^2y}{dx^2} + 0 + 2\left(\frac{dy}{dx}\right)^2 + 2f\left(\frac{dy}{dx}\right)^2 = 0$

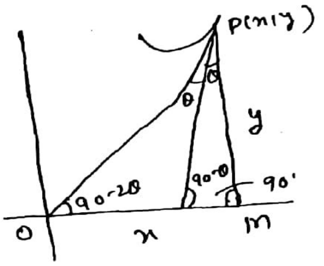
$1 + yy'' + y'^2 + f\cdot y'^2 = 0$

$yy''' + y'y'' + y'^3 + f\cdot y'^3 = 0$

$1 + yy'' + y'^2 + \frac{1}{y'}\left(-yy''' + y'y'' + y'^3\right)$

$\boxed{1 + yy'' + y'^2 - \left(\frac{yy'''}{y'} + y'' + y'^2\right) = 0.}$

Example:- A curve γ in the xy-plane such [illegible]
Joining the origine at the point $P(x,y)$ on the
curve and [illegible] parallel to y-axis through
P are equally inclined to the tangent to the
curve at the point P Then Find the D.E. of γ.



$$\tan(90-2\theta) = \frac{y}{x}.$$

$$\cot 2\theta = \frac{y}{x}.$$

$$\tan 2\theta = \frac{x}{y}. \quad -(\text{I})$$

$$\tan\theta = \frac{dy}{dx}. \quad -(\text{II})$$

from (I)

$$\frac{2\tan\theta}{1-\tan^2\theta} = \frac{x}{y}$$

$$\frac{2\cdot dy/dx}{1-(dy/dx)^2} = \frac{x}{y}.$$

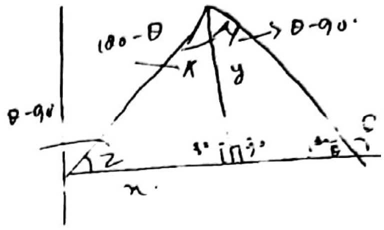
$$\boxed{2y\frac{dy}{dx} = x - x\left(\frac{dy}{dx}\right)^2}$$

which is required the solution.

Example:- Let $G = \{(x, f(x)) : 0 \le x \le 1\}$ be a real valued continuous and differentiable function.
Assume that $[0,1] \in G$.
Suppose that the tangent vector to G at any point in $\perp$ to the radius vector at that point which of the following is true

(I) Ellipse (II) parabola (III) circle (IV) line segment.

$$\frac{dy}{dx} = \tan\theta. \quad -(1)$$



$y + 90 + 180^\circ - \theta = 180^\circ$

$y = \theta - 90^\circ$

$x + y = 90$

$x + \theta - 90^\circ = 90^\circ$

$x = 180^\circ - \theta$

$z + 90^\circ + 180^\circ - \theta = 180^\circ$

$z = \theta - 90^\circ$

$\tan(\theta - 90^\circ) = \frac{y}{x}.$

$-\tan(90^\circ - \theta) = \frac{y}{x}.$

$-\cot\theta = \frac{y}{x}.$

$\tan\theta = \frac{-x}{y}$

$\frac{dy}{dx} = \frac{-x}{y}$

$y\,dy = -dx$

$\frac{y^2}{2} + \frac{x^2}{2} = c^2 \quad \Rightarrow \quad \frac{y^2}{2} + \frac{x^2}{2} = \frac{1}{2}$

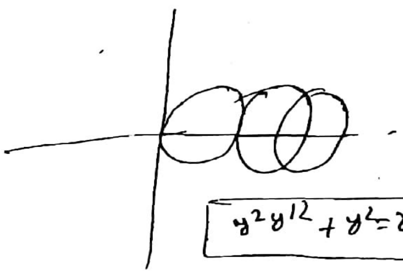
$\boxed{x^2 + y^2 = 1}$

$\frac{1}{2} + 0 = c^2$

Example:- Let γ be the family of all circle in xy-plane with radius r and centre at x-axis. find the differential to γ.

$(x-a)^2 + y^2 = r^2$

$2(x-a) + 2y\frac{dy}{dx} = 0$



$\boxed{y^2 y'^2 + y^2 = r^2}$

Order:-

The order of the higest derivative involved in the differential equation is called the order of the differential equation.

Example. $y = Ae^{Bx+C}$.

what is the minimum order of corresponding D.E..

(I) 1, (II) 2 ✓ (III) 3 (IV) 4.

$y = Ae^{Bx+C}$

$= Ae^{C} \cdot e^{Bx}$.

$y = De^{Bx}$, —(I) where $D = Ae^{C}$ = constant

$y' = DBe^{Bx}$.

$y' = B \cdot De^{Bx}$ —(II)

$y' = By$.

$y'' = B^2 De^{Bx}$ —(III)

$y'' = B^2 y$.

$y'' = \left(\frac{y'}{y}\right)^2 \cdot y$

$y'' = \frac{y'^2}{y}$

$\boxed{yy'' - y'^2 = 0}$

Note :- No of Independent arbitrary constant = order of the Differential. Eqn.

Example:- $y = C_1 + C_2\cos^2 x + C_3\sin^2 x + C_4\sin 4x$.

(I) 1 (II) 2 (III) 3 ✓ (IV) 4.

$y = c_1 + c_2 \cos^2 x + c_3 \sin^2 x + c_4 \sin 4x.$

$= c_1 + c_2 (1 - \sin^2 x) + c_3 \sin^2 x + c_4 \sin 4x.$

$= (c_1 + c_2) + (c_3 - c_2) \sin^2 x + c_4 \sin 4x.$

$y = A + B \sin^2 x + c_4 \sin 4x.$

$\Rightarrow$ Order of the diff eq^n = 3

Example:- $y = c_1 + c_2 \sin^2 x + c_3 \cos^2 x + c_4 \cos 2x.$

$= c_1 + c_2 \left(\frac{1 - \cos 2x}{2} \right) + c_3 \left(\frac{1 + \cos 2x}{2} \right) + c_4 \cos 2x.$

$= c_1 + \left(\frac{c_2 + c_3}{2} \right) + \left(\frac{c_3 - c_2}{2} + c_4 \right) \cos 2x.$

$y = A + B \cos 2x.$

(i) 1 (ii) 2 (iii) 3 (iv) 4

Degree of the D.E:- The power of the higest order derivative involving in it. When all the derivatives are free from radicals and fractional power.

OR.

The highest power of higest ~~derivati~~ order derivative provided all the derivative are in natural power.

Example:- $\frac{dy}{dx} = x^{1/3}$.

order = 1   degree = 1

Example:- $(y'')^{3/2} = (y'')^{2/3}$

$(y'')^9 = (y'')^4$ degree = 9, order 2.

$\frac{(y'')^9}{(y'')^4} = 1$

$(y'')^5 = 1$ $\cancel{\checkmark}$

Example:- $\frac{dy}{dx} = \sin x + y$

order = 1, degree = 1

$\sin\left(\frac{dy}{dx}\right) = y$.   polynomial.

order = 1, degree → Not find.

Linear and Non- Linear differential Equation

A differential equation of the form $f(x, y, y', y'', \dots y^n) = 0$ is called linear if.

(i) all the derivative and dependent variable is of degree 1.

(ii) There is no product b/w dependent variable and its derivative. (Not any transcendental function of dependent variable)

otherwise it is called non-Linear differential equation.

Note:

$a_0(x) y^n + a_1(x) y^{n-1} + \dots + a_{n-1} y' + a_n(x) y = Q(x)$ $\forall x$

where $a_0(x) \neq 0$, $a_1(x), \dots a_n(x)$ are continuous $\forall x$.

(ii) If $a_0(x), a_1(x), \dots a_n(x)$ all are constant then Differential eq^n is called linear diff^n eq^n with constant coefficient.

otherwise diff^n eq^n is called D.E. with variable coefficient.

(iii) If $Q(x) = 0$, $\forall x$ then Diff^n eq^n is called homogeneous L.D. equation other wise non-homogeneous L.D. eq^n.

Example:- $y''' + (y'')^{1/2} = 0$

$(y''')^2 + y'' = 0$ order $= 3$, degree $= 2$

Non-linear.

$y'' + \sqrt{y} = 0$ order $= 2$, degree $= 1$.

$y y'' + \log(xy) = 0$ order $= 2$, degree $= 1$

Non-linear.

Example:- Let $f(x, y, y', \dots y^n) = 0$ be the n^{th} order D.E. choose the incorrect.

(i) if $\deg f = 1 \Rightarrow f = 0$ is L.D.E.

(ii) if $f = 0$ is L.D.E $\Rightarrow \deg f = 1$. $\qquad \frac{dy}{dx} = \sin y$

(iii) if $\deg f > 1 \Rightarrow f = 0$ is Non-linear D.E.

(iv) if $f = 0$ is non-linear D.E. $\Rightarrow \deg f > 1$.

Solution of the differential Equation:-

Let $f(x, y, y', y'', \ldots y^n) = 0$ be the nth order D.E. define on the interval I. then the real (complex) valued function ϕ is called a solution define on I_0 if

(i) $\phi, \phi', \phi'', \ldots \phi^n$ exist

(ii) $\phi(x)$, satisfies the D.E. i.e.

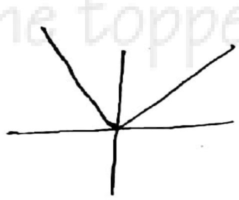
$$f(x, \phi, \phi', \phi'', \ldots \phi^n) = 0$$

Example:- Let $y = x|x|$, $\forall x \in \mathbb{R}$ be the solution of the D.E. $f(x, y, y', \ldots y^n) = 0$ Then the possible value of n is

(i) $n = 1$ (ii) $n = 2$ (iii) $n = 3$ (iv) $\forall n \in \mathbb{N}$.

soln:.

$y = |x|$



$y = x|x|$

$$y = \begin{cases} -x^2 & x < 0 \\ x^2 & x \geq 0 \end{cases}$$

$$y' = \begin{cases} -2x & x < 0 \\ 2x & x \geq 0 \end{cases}$$

$$L.H.D = \lim_{h \to 0} \frac{\phi(0+h) - \phi(h)}{h + 0}$$

$$= \lim_{h \to 0} \frac{-(2\cancel{h} - 0) - 0}{-h}$$

$$= \lim_{h \to 0} -2 = -2$$

$$R.H.D = \lim_{h \to 0} \frac{\phi(0+h) - \phi(0)}{h}$$

$$= \lim_{h \to 0} \frac{2h - 0}{h} = 2$$

then $y(t)$ is.

(i) $y(t)$ is diff^n but ... order derivative does not exist

(ii) $y(t)$ is twice differentiable but 3^{nd} order derivative does not exist

(iii) $y(t)$ is thrice diff^n but highest order derivative does not exist

(iv) $y(t)$ is infinite time differentiable.

$y' = y^2 + t \Rightarrow f(t, y) = y^2 + t$
$\Downarrow$
Polynomial function,
$\Rightarrow$ continuous & Differentiable.

$y'' = 2y \cdot y' + 1$

$y''' = 2y y'' + 2(y')^2 + 0$

$\vdots$

$y'' = 2y(y^2 + t) + 1$

$y'' = 2y^3 + 2yt + 1$

$y''' = 6y^2 + 2ty' + 2y$

infinitely ntimes differentiable.

Note :-

1. if $I_0 \subset I$ then ϕ is called local solution.

if $I_0 = I$ then ϕ is called global solution.

General solution:- A solution of the differential Equation in which No. of arbitrary constant is equal to order of the differential, is called general solution

$$(x-a)^2+y^2=9.$$

$$\Rightarrow \quad y^2(y'^2+1)=9.$$

Particular solution:- A solution of the differential Eq^n obtained from the general sol^n by taking some particular values of all arbitrary constant is called particular solution.

$$(x-a)^2+y^2=9.$$

$$(x-2)^2+y^2=9$$

$$(x-3)^2+y^2=9.$$

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Singular solution:- A solution of the Diff^n eq^n which is neither general sol^n nor particular solution is called singular solution.

A solution of the Diffⁿ eqⁿ which is envelope to the general solution is called the singular solution.

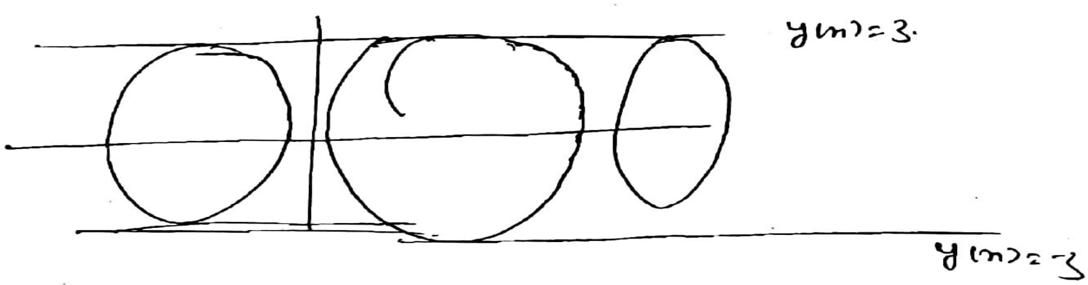
Envelope:- A curve $g(x,y)=0$ is called envelope to the curve $f(x,y,a)=0$ if

(i) corresponding to every point P on $g(x,y)=0$ there exist exactly one member of this family $f(x,y,a)=0$, which passes through P.

(ii) Every member of this family $f(x,y,a)=0$ passes through exactly one point of $g(x,y)=0$

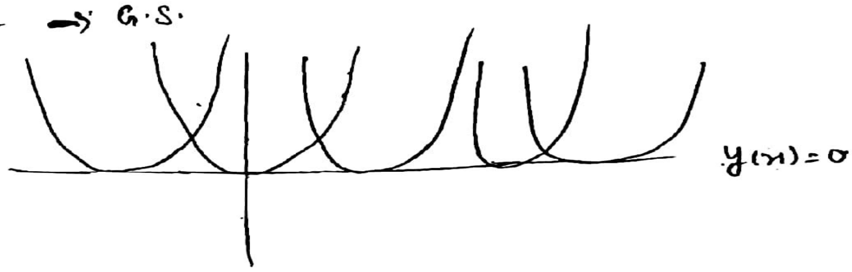
Example:-

$y(x)=3,$
$y'(x)=0$
$y^2(y'^2+1)=9$
$9\cdot(0+1)=9.$



$\Rightarrow$ $y(x)=3$ and $y(x)=-3$ is singular solution

$y=(x-a)^2 \rightarrow$ G.S.



Note :- Singular solution is exist in non-linear diff^n equation.

Example :- $x\,dy - y\,dx = 0, \quad x \geq 0, \quad y(0) = 0$

$x\frac{dy}{dx} - y = 0 \quad \Rightarrow \boxed{y(x) = 0, \ \forall x \geq 0}$

Now

$x\frac{dy}{dx} - y = 0$

$\frac{dy}{y} = \frac{dx}{x}, \quad x \neq 0, \ y(x) \neq 0$

$y > 0 \quad x > 0 \quad c > 0$

$\log y = \log x + \log c$

$y = cx, \quad G.S \ — \ x > 0$

$f(x) = c \quad \Rightarrow f'(x) = 0$

or $f'(x) = 0 \quad \Rightarrow f(x) = c \quad$ (True / false.)

Example:

$$f(x) = \begin{cases} 1 & : x \in (0,1) \\ 2 & : x \in (2,3) \end{cases}$$

\# $y(x)$ be the differentiable fn.

$y(x) = c \quad \Rightarrow y'(x) = 0$

or $y'(x) = 0 \quad \Rightarrow y(x) = c.$

constant function:-

A function is constant if range set has exactly one element.

Non-constant function :

A function is non-constant function iff range set has at least two elements.

connected domain:-

it is singlton (interval).

Example:- Let $f: I_1 \cup I_2 \longrightarrow \mathbb{R}$. is such that $I_1 \cap I_2 = \phi$ and $f'(x) = 0 \quad \forall x$

Then range f has.

(i) at least two elements.

(ii) at most two points

(iii) exactly two points.

(iv) f is constant.

$$f(x) = \begin{cases} 1 & : x \in I_1 \\ 2 & : x \in I_2 \end{cases} \Rightarrow f'(x) = 0 \quad \forall x.$$

$$f(x) = \begin{cases} 1 & : x \in I_1 \\ 1 & : x \in I_2 \end{cases}$$

$$\Rightarrow f'(x) = 0 \quad \forall x \in I_1 \cup I_2$$