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MATHEMATICAL SCIENCE

VOLUME - VI

SIERRA INNOVATIONS

INDEX

Partial Differential Equation (P.D.E.)

Partial Differential Equation:-

A Equation which contains P-Derivative of dependent variable w.r. to two or more than two independent variable is called Partial differential equation.

Example:- $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = x+y$, $x^2 \frac{\partial^2 z}{\partial x^2} + y \frac{\partial^2 z}{\partial x \partial y} = xyz^2$

Degree of P.D.E:

classification of first order P.D.E.

Linear Partial D.E : A first order partial differential Eqn is said to be Linear if it is Linear p, q and z and of the form

$$P(x,y)\,p + Q(x,y)\,q = R(x,y)z + S(x,y)$$

$$p = \frac{\partial z}{\partial x}, \quad q = \frac{\partial z}{\partial y}, \quad r = \frac{\partial^2 z}{\partial x \partial y}$$

Example:- $p + q = x+y$

$$y^2 p + xq = xyz + y^2$$

If $S(x,y) = 0$ then it is called Homogeneous Linear partial Differential equation.

if $S(x,y) \neq 0$ then it is called non-Homogeneous P.D.E.

Semi Linear P.D.E: A first order partial differential Eqn is said to be semilinear if it is linear and of p and q but not necessary in z.

The form.

$$P(x,y)\,p + Q(x,y)\,q = R(x,y,z)$$

Example:- $x^2p + y^2q = xyz$

$p + q = z^2.$

Quasi Linear P.D.E
A first order P.D.E is said to be quasi linear if it is linear p and q and of the form $P(x,y,z)\,p + Q(x,y,z)\,q = R(x,y,z)$.

Example:- $yz\,p + (z-x)\,q = x-y.$

$xp + yq = z^2.$

L ⊆ semilinear ⊆ quasi Linear

Non-Linear P.D.E :-
A first order P.D.E. is said to be non-linear if it does not come under any one of the above kind.

Example:- $p^2 + q^2 = 1.$

$pq = z.$

Formation of First order partial D.E:-

- By elimination of arbitrary constant
- By elimination of arb^r function.

1. **By Elimination of arbitrary constant:-**
Let ~~x,y,z and as b~~ $F(x,y,z,a,b) = 0$
where a and b are arbitrary constant and x, y are independent z is dependent variable.

Differential partially w.r. to x & y.

$F_1(x,y,z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}) = 0$ — (II)

$F_2(x,y,z, \frac{\partial z}{\partial y}, a, b) = 0$ — (III)

eliminating a and b from eqⁿ (I) & (II),

we get

$\phi(x,y,z,p,q) = 0$

Example:- P.D.E representing the set of all sphere of unit radius with centre in x-y-plane is

(I) $1+p^2+q^2=0$ (II) $py-qx=0$

(III) $z^2(1+p^2+q^2) = 1$ (IV) N.O.T.

$(x-a)^2+(y-b)^2+z^2=1$

$2(x-a) + 2z\frac{\partial z}{\partial x} = 0 \Rightarrow x-a = -zp$

$2(y-b) + 2z\frac{\partial z}{\partial y} = 0 \Rightarrow y-b = -zq$

$z^2p^2+z^2q^2 = 1-z^2$

$\boxed{z^2(p^2+q^2+1) = 1}$

Example: The P.D.E $z^2(1+a^3) = 8(x+ay+b)^3$ is

(I) $z = p^2+q^2$ (II) $27z = p^2+q^2$

(III) $27z = p^3+q^3$ $z = p^3+q^3$

$2zp(1+a^3) = 24(x+ay+b)^2$

$2zq(1+a^3) = 24a(x+ay+b)^2$

$\frac{p}{q} = \frac{1}{a} \Rightarrow a = \frac{p}{q}$

$z^2\left(1+\frac{p^3}{q^3}\right) = 8\ (x+\frac{p}{q}$ ·

$8\left(\cancel{x}+\frac{by}{q}+6b-\cancel{x}+\frac{by}{q}\right)^3$

$= 8\left(\frac{2by}{q}+6b\right)^3$

$z^2(q^3+p^3) = \frac{8(2by+6b)^3}{6}$

$\frac{z}{2b} = \frac{1}{3}\ (x+ay+b)$

$\frac{3z}{2b} = (x+a+b)$

$\frac{3\cancel{z}}{\cancel{2}} = \frac{z^{\cancel{2}}}{\cancel{8}\,4}(1+a^3)$

$3 = z\,($

Example:- $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

$\frac{1}{a^2}x + \frac{z}{c^2}p = 0$

$\frac{1}{a^2} + \frac{1}{c^2}[zr + p^2] = 0$

$\frac{1}{a^2} = -\frac{1}{c^2}[zr + p^2]$

$-\frac{x}{c^2}[zr + p^2] + \frac{z}{c^2}p = 0$

$-x[zr + p^2] + zp = 0$

$-xzr - xp^2 + zp = 0$

$$\boxed{xzr + xp^2 = zp}$$

$\frac{y}{b^2} + \frac{z}{c^2}q = 0$

$\frac{1}{b^2} + \frac{1}{c^2}[zr + q^2] = 0$

$-\frac{1}{c^2} \cdot y + [zr + q^2] + \frac{zq}{c^2} = 0$

$$\boxed{yzr + yq^2 = zq}$$

Similarly

$zs + pq = 0.$

Note:- By elimination of arb^r constant we can get both non-linear as well as quasi linear.

(b) If No. of arb^r constant = No. of independent variable Then will be get unique P.D.E.

(c) No. of arb^r const is less then No. of independent variable then we can have more than one P.D.E.

(d) No. of arb^r constant is greater than No. independent variable we can have usually P.D.E. of order greater one.

Example: $\log(az-1) = x+ay+b.$

$$az-1 = e^{x+ay+b}.$$

$$a\frac{\partial z}{\partial x} = e^{x+ay+b}$$

$$ap = e^{x+ay+b}.$$

$$\frac{1}{az-1}\cdot ap = 1.$$

$$ap = az-1$$

$$a(p-z) = -1$$

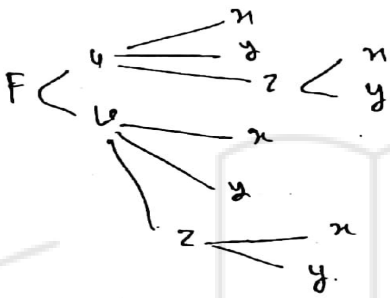
$$a = \frac{-1}{p-z}$$

By Ellimination of arbitrary function:-

Suppose u and v are any two function of x, y and z.

Let F be an arbitrary function of u and v and of the form $F(u,v)=0$ OR

$u = f(v)$ or $v = F(u)$.



Differentiate partially w.r.t. x and y.

$$\frac{\partial F}{\partial u}\left[\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial x}\right]+\frac{\partial F}{\partial v}\left[\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\cdot\frac{\partial z}{\partial x}\right]=0$$

$$\frac{\partial F}{\partial u}\left[\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial y}\right]+\frac{\partial F}{\partial v}\left[\frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}\cdot\frac{\partial z}{\partial y}\right]=0$$

$$\frac{\partial F}{\partial u}\left[\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}\frac{\partial z}{\partial x}\right]=-\frac{\partial F}{\partial v}\left[\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\cdot\frac{\partial z}{\partial x}\right]$$

$$\frac{\partial F}{\partial u}\left[\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial y}\right]=-\frac{\partial F}{\partial v}\left[\frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}\cdot\frac{\partial z}{\partial y}\right]$$

$$\frac{\frac{\partial F}{\partial u}\left[\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial x}\right]}{\frac{\partial F}{\partial u}\left[\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\frac{\partial z}{\partial y}\right]}=\frac{-\frac{\partial F}{\partial v}\left[\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\cdot\frac{\partial z}{\partial x}\right]}{-\frac{\partial F}{\partial v}\left[\frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}\cdot\frac{\partial z}{\partial y}\right]}$$

$$\frac{\left[\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial x}\right]}{\left[\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial y}\right]}=\frac{\left[\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\cdot\frac{\partial z}{\partial x}\right]}{\left[\frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}\cdot\frac{\partial z}{\partial y}\right]}$$

$[u_x \cdot v_y + u_z v_y q] p + u_x v_z \cdot q + u_z u_z z - \dots$

$+ u_y v_z p + u_z u_z$

$$\boxed{\frac{\partial(u,v)}{\partial(y,z)} p + \frac{\partial(u,v)}{\partial(z,x)} \cdot q = \frac{\partial(u,v)}{\partial(x,y)}}$$

$P(x,y,z) p + Q(x,y,z) q = R(x,y,z).$

$$\begin{vmatrix} p & q & -1 \\ u_x & u_y & u_z \\ v_x & v_y & v_z \end{vmatrix} = 0$$

Example:- Find the P.D.E. by Eliminating the arb^r function $F(x+y+z,\ x^2+y^2-z^2) = 0$

$u = x+y+z, \quad v = x^2+y^2-z^2 =$

$$\begin{vmatrix} p & q & -1 \\ 1 & 1 & 1 \\ 2x & 2y & -2z \end{vmatrix} = 0$$

$p(-2z-2y) - q(-2z-2x) - 1(2y-2x) = 0$

$(z+y) p - (z+x) q = (x-y)$

which is required the solution.

Example:- Find the P.D.E. by Eliminating the arb^r function. $z = xy + f(x^2+y^2)$

$z = xy + f(x^2+y^2)$

$p = y + f'(x^2+y^2) \cdot 2x.$

$q = x + f'(x^2+y^2) \cdot 2y$

$yp - xq = y^2 - x^2$

which is required the solution.

$z = xy + f(x^2+y^2)$

$z - xy = f(x^2+y^2)$

$u = z - xy, \quad v = x^2+y^2.$

$$\begin{vmatrix} p & q & -1 \\ -y & -x & 1 \\ 2x & 2y & 0 \end{vmatrix} = 0$$

$p(-2y) - q \cdot (+2x) - 1(-2y^2+2x^2) = 0$

$-y \cdot p + q \cdot x = x^2 - y^2$

$yp - qx = y^2 - x^2$

Example:- $z = f(\frac{xy}{2})$

$p = f'(\frac{xy}{2}) \cdot \frac{y}{2}$

$pz = f'(xy) \cdot y \quad$ —(i)

$q = f'(xy) \frac{x}{2}$

$qz = f'(xy) \cdot x \quad$ —(ii)

$\frac{pz}{qz} = \frac{y}{x}$

$\boxed{px = qy}$

Example:- $z = y^2 + 2f\left(\frac{1}{x} + \log y\right)$

Ans = $px^2 + qy = 2y^2$

$f(x^2+y^2+z^2, x^2+y^2+xy) = 0$

Ans - $z(x+2y)p - z(y+2x)q = y^2 - x^2$

$z = e^{ny} f(x-y)$.

$p + q = nz$

Example:- $z = f(x+iy) + g(x-iy)$, $i = \sqrt{-1}$

Let $a = i$

$z = f(x+ay) + g(x-ay)$.

$p = f'(x+ay) + g'(x-ay)$

$q = af'(x+ay) - ag'(x-ay)$

$r = f''(x+ay) + g''(x-ay)$

$t = a^2 f''(x+ay) + a^2 g''(x-ay)$

$t = a^2 r$.

$\boxed{r+t=0,}$ $\quad a = i$

which is required the solution of the given equation problem.

Example:- $u(x,y) = f(x, e^y) + g(y^2 \cos y)$ then corresponding P.D.E.

(i) $u_{xy} + x u_{xx} = u_x$

(ii) $u_{xy} + x u_{xx} = x u_x$

(iii) $u_{xy} - x u_{xx} = u_x$

(iv) $u_{xy} - x u_{xx} = x u_x$

$u(x,y) = f(xe^y) + g(y^2 \cos y)$

$u_x = f'(xe^y) e^y +$

$u_{xy} = x f''(xe^y) e^{2y} + f'(xe^y) e^y$

$u_{xx} = f''(xe^y) e^{2y} -$

$u_{xy} = x u_{xx} + u_x$

$u_{xy} - x u_{xx} = u_x$

Example:- $x = f(z) + g(y)$ Then corr. P.D.E. if

(i) $p + r = 0$ (ii) $q + t = 0$ (iii) $ps = qr$

(iv) $p = q$.

$$x = f(z) + g(y).$$

$$p = f'(z)\cdot\frac{\partial z}{\partial x} \Rightarrow p f'(z) = 1$$

$$0 = f'(z)\frac{\partial z}{\partial y} + g'(y).$$

$$= f' q + g'(y)$$

$$0 = \cancel{f''(z)\frac{\partial^2 z}{\partial x^2}} + f'(z)\, r = p^2 f'' + f'(z)\cdot r$$

$$0 = f''\frac{\partial z}{\partial x}\cdot\frac{\partial z}{\partial y} + f''\frac{\partial^2 z}{\partial x \partial y}$$

$$= f''\, p\cdot q + f'\cdot s$$

$$p^2 f'' + f'(z)\cdot r = 0$$

$$p^2 f'' = -f'(z)\cdot r$$

$$pq f'' = -f'(z)\cdot s$$

$$\frac{p^2}{pq} = \frac{r}{s}$$

$$ps = q\cdot r$$

Note:- By elimination of one arbitrary funn, we will always get quasi linear P.D.E.

No. of arbr function in equation is equal to order of the corresponding partial D.E

Integral surface:

Let $Pp + Qq = R$ —(i) be the quasi linear P.D.E.

Let $z = z(x, y)$ —(ii) be the solution of equation 1.

$$Pz_x + Qz_y = R$$

$$Pz_x + Qz_y - R = 0$$

$$(P, Q, R) \cdot (z_x, z_y, -1) = 0$$

$$(P, Q, R)\ (p, q, -1) = 0 \quad \text{—(iii)}$$

∴ vector $(p, q, -1)$ in ⊥ to eqn (ii)

∴ solution eqn should be satisfy the eqn (iii).

The vector P, Q, R is tangent to every point of the surface. then this kind the surface is called integral surface.

In order to find the integral surface or characteristic curve.

$$x = f_1(t), \quad y = f_2(t), \quad z = f_3(t)$$

$\left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right)$ tangent vector

$$\frac{dx}{dt} = P, \quad \frac{dy}{dt} = Q, \quad \frac{dz}{dt} = R$$

$$\frac{dx}{P} = dt, \quad \frac{dy}{Q} = dt \quad \frac{dz}{R} = dt$$

$$\boxed{\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}}$$

This is known as Lagrange's Method. (L.A.E)

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General solution of the quasilinear P.D.E:-

Let $Pp + qQ = R$ be the quasi l. P.D.E

Let general solution of eq^n (1) is given by

$\phi(u, v) = 0$, $u = \phi(v)$, $v = \phi(u)$

ϕ is an arb^r function and $u = c_1$ $v = c_2$

be the solution of the lagrange A.E which is given by

$$\boxed{\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}}$$

u & v are independent solution at least one of u and v must contain z.

case-I Let $Pp + Qq = R$ —(i) be the quasi linear P.D.E.

$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ — (ii) be the Lagran. A.E.

take two different pair of fraction of eq^n (ii)

we get $u(x, y, z) = c_1$ $v(x, y, z) = c_2$.

Example:- $yzp + zxq = xy$.

$$\frac{dx}{yz} = \frac{dy}{zx} = \frac{dz}{xy}.$$

$$\frac{dx}{yz} = \frac{dy}{zx} \Rightarrow \frac{dx}{y} = \frac{dz}{x}.$$

$\Rightarrow x \cdot dx = y \cdot dz$

$\Rightarrow x^2 = y^2 + c_1$

$\Rightarrow x^2 - y^2 = c_1$

$\Rightarrow u = x^2 \; y^2 = c_1$

$\frac{dy}{zx} = \frac{dz}{xy}$

$\frac{dy}{z} = \frac{dz}{y}$

$y\,dy = z \cdot dz$

$y^2 - z^2 = c_2$

$v = y^2 - z^2 = c_2$.

(i) $\phi_1(x^2-y^2) = y^2-z^2$ $\qquad \phi_2(y^2-z^2, x^2-z^2)=0$

(ii) $\phi_2(x^2-y^2) = x^2-z^2$

(iii) $z^2 = y^2 - \phi(x^2-y^2)$.

Example:- $xp + yq = z$.

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}$$

$$\frac{dx}{x} = \frac{dy}{y} \qquad \frac{dy}{y} = \frac{dz}{z}, \qquad \frac{dx}{x} = \frac{dz}{z}$$

$\log x = \log y + \log c_1$, $\quad \log y = \log z c_2 \quad \log x = \log z c_3$

$x = y c_1$, $\quad y = z c_2 \quad x = z c_3$

$\phi\left(\frac{x}{y}\right) = \frac{y}{z}$, $\quad \phi\left(\frac{y}{z}\right) = \frac{x}{z}$, $\quad \phi\left(\frac{x}{y}\right) = \frac{x}{z}$

Example:- $y^2 p - xyq = x(z-2y)$

$\phi(x^2+y^2, y^2-yz) = 0$

$$\frac{dx}{y^2} = \frac{dy}{-xy} = \frac{dz}{x(z-2y)}$$

$\frac{dx}{y^2} = \frac{dy}{-xy}$ $\qquad \frac{dy}{-xy} = \frac{dz}{x(z-2y)}$

$x\,dx = -y\,dy$ $\qquad \frac{dz}{-y} = \frac{dz}{z-2y}$. $\qquad$ Hence,

$x^2+y^2 = c_1$ $\qquad \phi(x^2+y^2, y^2-zy) = 0$ Ans.

$$\frac{dz}{dy} = \frac{z-2y}{-y}$$

$$\frac{dz}{dy} = -\frac{z}{y} + 2.$$

$$\frac{dz}{dy} + \frac{z}{y} = -2.$$

$$I.F = e^{\int \frac{1}{y} dy} = e^{\log y} = y.$$

$$z \cdot y = \int +2y \cdot dy + c_2$$

$$zy = +y^2 + c_2 \qquad y^2 = zy + c_2$$