Toppersnotes

Unleash the topper in you

CSIR-NET

Council of Scientific & Industrial Research

MATHEMATICAL SCIENCE

VOLUME - VIII

SIERRA INNOVATIONS

INDEX

Continuity

Let $f: S \longrightarrow \mathbb{R}$.

$\alpha \in S$.

then f is continuous at $x = \alpha$.

if $\alpha \in S - S'$

but $\alpha \in S'$ as well.

then f is continuous at $x = \alpha$.

if $\lim_{x \to \alpha} f(x) = f(\alpha)$. i.e. $\lim_{x \to \alpha} f(x)$ exist

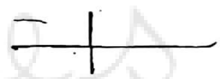
finitely and equal to the $f(\alpha)$.

Example :- $S = \{ \frac{1}{\sqrt{n}} : n \in \mathbb{N} \} \cup \{0\}$.

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ -1 & x \in \mathbb{Q}^c \end{cases}$$

$f: S \longrightarrow \mathbb{R}$

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ -1 & x \in \mathbb{Q}^c \end{cases}$$



if $\alpha \in S$, $\alpha \neq 0$

$\Rightarrow$ f is continuous at $x = \alpha$.

but if $\alpha = 0$

$\langle a_n \rangle \longrightarrow 0$

$\Rightarrow$ f is not continuous at $x = 0$.

Note : If we say f is continuous on $S \subseteq \mathbb{R}$ it means.

f is continuous at every point of in S.

2. (i) f_i, $i = 1$ to 9 is continuous on $\mathbb{R}$.

(ii) H is continuous on its <u>natural domain</u>.

i.e. $\mathbb{R} - Z(f_j)$.

3. Branch function is continuous at every non-connecting point but at connecting point will have to check.

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Example:

$$f(x) = \begin{cases} x \cdot \sin x & x<0 \\ 1-\cos x\pi & 0 \le x < 1 \\ 2x & 1 \le x. \end{cases}$$

$\Rightarrow$ f is not continuous at $x = 0, 1$.

4). <u>chicken pox.</u>:

$$ch(x) = \begin{cases} f_1(x) & x \in Q. \\ f_2(x) & x \in Q^c. \end{cases}$$

ch(x) is continuous at $x = \alpha$ iff

$f_1(\alpha) = f_2(\alpha)$.

<u>Example</u>:

$$f(x) = \begin{cases} 1-\cos x & x \in Q \\ \sin x & x \in Q^c. \end{cases}$$

f(x) is continuous at $x=\alpha$ iff $f_1(\alpha) = f_2(\alpha)$

$1-\cos\alpha = \sin\alpha$

$1 = \sin\alpha + \cos\alpha$

$\frac{1}{\sqrt{2}} \sin(x + \pi/4)$

$\sin(x + \pi/4) = \frac{1}{\sqrt{2}} = \sin \pi/4$

$x + \pi/4 = 2n\pi + \pi/4$

$\boxed{x = 2n\pi}$, $n \in Z$

$\Rightarrow$ f(x) is continuous at $x = 2n\pi$.

(v) Small pox function:-

$$f(x) = \begin{cases} \frac{1}{|p|+1} + \cos\frac{1}{q} & x = p/q \\ \cos 2\pi x & x \in \mathbb{Q}^c \end{cases}$$

$$g(x) = \begin{cases} \frac{1}{|p|+1} + \cos\frac{1}{q} & x = p/q \\ \cos 2x & x \in \mathbb{Q}^c. \end{cases}$$

DoC of f and g are.

$\lim_{x\to 0} f(x)$ and $\lim_{x\to 0} g(x)$ does not exist

$\Rightarrow$ f and g are not continuous at $x = 0$.

$\langle a_n \rangle \to \alpha \neq 0$

$$f(a_n) = \begin{cases} \frac{1}{|p_n|+1} + \cos\frac{1}{q_n} & x = \frac{p_n}{q_n} \\ \cos 2\pi a_n & a_n \in \mathbb{Q}^c \end{cases}$$

$$= \begin{cases} 1 & x \in \mathbb{Q}\ p/q \\ \cos 2\pi\alpha. & x \in \mathbb{Q}^c \end{cases}$$

$\lim_{x\to\alpha} f(x)$ exist iff $\cos 2\pi\alpha = 1 = \cos 2n\pi$

$2\pi\alpha = 2n\pi$

$\alpha = n, \quad n \in \mathbb{Z}-\{0\}$

$\Rightarrow$ f(x) is not continuous at $x = \alpha \in \mathbb{Z}$

$\langle a_n \rangle \to \alpha \neq 0$

$$g(a_n) = \begin{cases} \frac{1}{|p_n|+1} + \cos\frac{1}{q_n} & a_n = \frac{p_n}{q_n} \\ \cos 2\alpha. & a_n \in \mathbb{Q}^c. \end{cases}$$

$$= \begin{cases} 1 & x \in \mathbb{Q} \\ \cos 2\alpha. & x \in \mathbb{Q}^c \end{cases}$$

$\lim_{x \to \alpha} f(x)$ exist iff $\cos 2\alpha = 1 = \cos 2n\pi$

$$2\alpha = 2n\pi$$

$$\alpha = n\pi$$

$\lim_{x \to \alpha} f(x)$ is continuous at $x = \alpha$ iff

$$\lim_{x \to \alpha} f(x) = f(\alpha), \quad \alpha \in \mathbb{Q}^c.$$

$\Rightarrow$ $f(x)$ is continuous at $x \neq n\pi, \; n \in \mathbb{Z} - \{0\}$.

Example:- $f(x) = \begin{cases} 1/q & x = p/q \\ 0 & x \in \mathbb{Q}^c. \end{cases}$

$\langle a_n \rangle \to \alpha.$

$f(a_n) = \begin{cases} \frac{1}{q_n} & a_n = \frac{p_n}{q_n} \Rightarrow 0 \\ 0 & a_n \in \mathbb{Q}^c \Rightarrow 0 \end{cases}$

$\lim_{x \to \alpha} f(x)$ exist iff $0 = 0$

$\Rightarrow$ $f(x)$ is continuous at $x = \alpha \in \mathbb{Q}^c$

iff $\lim_{x \to \alpha} f(x) = f(\alpha), \quad \alpha \in \mathbb{Q}^c$

Example:- $f(x) = \begin{cases} \frac{1}{q} & x = p/q \\ \frac{1}{q^2} & x = \sqrt{p/q} \\ 0 & \text{else.} \end{cases}$

$\langle a_n \rangle \to \alpha,$

$f(a_n) = \begin{cases} \frac{1}{q_n} & x = \frac{p_n}{q_n} \\ \frac{1}{q_n^2} & x = \sqrt{\frac{p_n}{q_n}} \\ 0 \end{cases}$

$\lim_{x\to\alpha} f(x)$ exist at $x = \alpha \in \mathbb{R}$.

$f(x)$ is continuous at $x = \alpha$ iff $\lim_{x\to\alpha} f(x) = f(\alpha)$

$\Rightarrow \alpha \in \mathbb{Q}^c \cup \{\sqrt{p}/q\}$.

Note :- limit at ∞.

$$\lim_{x\to\infty} f(x) = \lim_{x\to 0^+} f\left(\frac{1}{x}\right)$$

$$\lim_{x\to -\infty} f(x) = \lim_{x\to 0^-} f(1/x).$$

Example:– $f(x) = \frac{\sin x}{x}$, $x \neq 0$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{\sin x}{x} = \lim_{x\to 0^+} \sin(1/x)\cdot x.$$

$$= 0$$

$\Rightarrow f(x)$ exist at $x = \infty$.

Example:- $f(x) = x\cdot\sin 1/x$.

$$\lim_{x\to\infty} f(x) = \lim_{x\to 0^+} \frac{1}{x}\cdot\sin x.$$

$$= \lim_{x\to 0} \frac{\sin x}{x} = 1.$$

Results and Properties

1. Algebra of continuity is valid.

$\rightarrow$ L. combination, product, max, mini, f/g. are continuous..

2. f and g are two function and $g(x)$ is continuous at $x=\alpha$ and f is continuous at $x=g(\alpha)$. then $(f\circ g)$ is continuous at $x=\alpha$.

3. $|f(x)| \le |x|$, $x \in \mathbb{R}$.

then $\lim_{x\to 0} f(x) = 0 = f(0)$.

$\Rightarrow$ $f(x)$ is continuous at $x=0$

$|f(x)-0| \le |x-0| < \epsilon = \delta$.

Imp

Cauchy functional Equation:-

If $f: \mathbb{R} \to \mathbb{R}$ such that

$f(x+y) = f(x) + f(y)$, $\forall x, y \in \mathbb{R}$

we say f hold C.F.E.

1. $f(0)=0$, $f(-x) = -f(x)$.

2. $f(mx) = m f(x)$, $\forall m \in \mathbb{Z}$, $\forall x \in \mathbb{R}$.

3. $f(rx) = r f(x)$ $\forall r \in \mathbb{Q}$, $x \in \mathbb{R}$.

4. $f(m) = f(1+1+1+ \cdots +1)$, $m \in \mathbb{N}$.

$= f(1) + f(1) + \cdots + f(1)$

$= m \cdot f(1)$.

$f(-m) = -m f(1)$.

$\Rightarrow f(n) = cn, \quad f(1) = c, \quad \forall n \in \mathbb{Z}$

$\Rightarrow f(\frac{1}{q} + \frac{1}{q} + \cdots + \frac{1}{q}) = f(1) = c$

q - times.

$q \cdot f(\frac{1}{q}) = c.$

$f(\frac{1}{q}) = \frac{1}{q} \cdot c.$

$f(p/q) = f(\frac{1}{q} + \frac{1}{q} + \cdots + \frac{1}{q})$

p-time.

$= p\, f(\frac{1}{q}) = p \cdot \frac{1}{q} \cdot c$

$f(p/q) = c \cdot p/q.$

$f(-p/q) = -f(p/q) = -c\, p/q$

$= (-p/q) \cdot c$

5. $f(n) = cx, \quad \forall n \in \mathbb{Q}.$

$$f(x) = \begin{cases} \frac{1}{q} & x = p/q \\ 0 & \text{else.} \end{cases}$$

Let $\alpha \in \mathbb{Q}^c$

$\Rightarrow f(\alpha) = 0$

Now for given $\epsilon > 0$

$\exists k \in \mathbb{N}.$

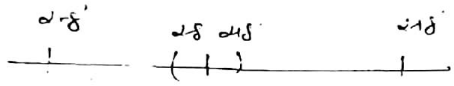
s.t $\frac{1}{n} < \epsilon \quad \forall n \geq k.$

for any $\delta' > 0$ in $(\alpha - \delta', \alpha + \delta')$

there are only finite rational.

viz $t_1, t_2, \ldots, t_m.$

s.t. $t_i = \frac{p_i}{q_i}$ and $q_i < k$



$\delta_r = |t_r - \alpha|$

$\min\{\delta_1, \delta_2, \dots, \delta_m\} = \delta$

$\Rightarrow t_r \notin (\alpha-\delta, \alpha+\delta)$

if $x \in (\alpha-\delta, \alpha+\delta) \Rightarrow x = p/q$, $q \geq k$.

$|f(x) - f(\alpha)| = 0$ $\quad x \in \mathbb{Q}^c$

$= \frac{1}{q} < \epsilon$ on $q \geq k$.

$|f(x) - f(\alpha)| < \epsilon$ $\forall x \in (\alpha-\delta, \alpha+\delta)$

$\Rightarrow$ f is continuous at $x = \alpha \in \mathbb{Q}$.

α is arbitarary

$\Rightarrow$ f is continuous on $\mathbb{Q}^c$.

Let $f(x+y) = f(x) + f(y)$, $\forall x, y \in \mathbb{R}$.

if f is continuous on $\mathbb{R}$

then $f(x) = cx$, $\forall x \in \mathbb{R}$, $c = f(1)$

as f holds C.F.E.

$\Rightarrow f(x) = cx$, $\forall x \in \mathbb{Q}$.

Let $\alpha \in \mathbb{R} \Rightarrow$ f is continuous at $x = \alpha$.

if $\langle a_n \rangle \to \alpha$. $\Rightarrow \langle f(a_n) \rangle \longrightarrow \langle f(\alpha) \rangle$.

if $\langle a_n \rangle \to \alpha$, $a_n \in \mathbb{Q}$.

$\langle f(a_n) \rangle = \langle c \cdot a_n \rangle \to c \cdot \alpha$.

$\to f(\alpha)$.

$\Rightarrow f(\alpha) = c\alpha$.

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$\Rightarrow f(x) = \alpha x$

as α is an arbitrary.

\# condition for C.F.E to be continuous on R.

if $f(x+y) = f(x) + f(y)$, $\forall x, y \in \mathbb{R}$ Then in any one of the following conditions ~~f~~ is satisfy f is continuous on $\mathbb{R}$.

1. f is continuous at $x=0$.
2. $|f(x)| \leq |x|$
3. f is continuous at one point
4. f is monotonic.
5. $x>0, \Rightarrow f(x) \geq 0$.
6. $f(x \cdot y) = f(x) \cdot f(y)$, $\forall x, y \in \mathbb{R}$.
7. f is bounded on (a, b) given $a \neq b \in \mathbb{R}$.

<u>Proof</u> :-

f is continuous at $x=0$

$$\lim_{h \to 0} f(0+h) = 0 = f(0)$$

$$\lim_{h \to 0} f(0-h) = 0 = f(0)$$

let $\alpha \in \mathbb{R}$

$$\lim_{h \to 0} f(\alpha+h) = \lim_{h \to 0} [f(\alpha) + f(h)]$$

$$= \lim_{h \to 0} f(\alpha) + \lim_{h \to 0} f(0+h)$$

$$= f(\alpha)$$

$$\lim_{h \to 0} f(\alpha-h) = \lim_{h \to 0} [f(\alpha) + f(-h)]$$

$$= \lim_{h \to 0} f(\alpha) + \lim_{h \to 0} f(0-h)$$

$= f(\alpha)$

$\Rightarrow$ f is continuous at $x = \alpha$

α is arbitrary.

$\Rightarrow$ f is continuous on $\mathbb{R}$.

f is continuous at $x = \alpha$.

$\langle a_n \rangle \to \alpha$

$b_n = a_n - \alpha$.

$\langle b_n \rangle \to 0$

$f(b_n) = \cancel{\lim} f(a_n - \alpha) \rangle = \langle f(a_n) \rangle - \langle f(\alpha) \rangle$

$= f(\alpha) - f(\alpha)$

$= 0 = f(0)$

$\Rightarrow$ f is continuous at $x = 0$

$\Rightarrow$ f is continuous on $\mathbb{R}$

4 f is monotonic $\Rightarrow$ f is continuous on $\mathbb{R}$

W.L.G. $f\uparrow$.

$\langle a_n \rangle \to 0$

$|a_n| < 1/n$

$-1/n \le a_n \le 1/n$

$f(-1/n) \le f(a_n) \le f(1/n)$

$-\frac{1}{n} \cdot c \le f(a_n) \le \frac{1}{n} \cdot c$

$\Rightarrow |f(a_n)| \le \frac{|c|}{n}$.

$\Rightarrow \langle f(a_n) \rangle \to 0$

5 $x > 0,\ f(x) \geq 0$

Let $x_1 < x_2$

$\Rightarrow x_2 - x_1 \geq 0$

$\Rightarrow f(x_2 - x_1) \geq 0$

$\Rightarrow f(x_2) - f(x_1) \geq 0$

$\Rightarrow f(x_1) \leq f(x_2)$

$\Rightarrow f \uparrow$

6. $f(x \cdot y) = f(x) \cdot f(y)$

$a > 0$

$\Rightarrow \delta = \sqrt{a}$, is define

$\Rightarrow \delta^2 = a$.

$\Rightarrow f(a) = f(\delta^2) = f(\delta) \cdot f(\delta) = (f(\delta))^2 \geq 0$

7. f is bounded on (a, b)

define $g(x) = f(x) - x f(1)$

$\Rightarrow g(x) = 0,\ \forall x \in \mathbb{Q}$

and $g(x+y) = \cancel{g(x) + g(y)}\ f(x+y) - (x+y) \cdot f(1)$

$= f(x) + g(x) - [x f(1) + y f(1)]$

$= f(x) - x f(1) + g(x) - y f(1)$

$\Rightarrow g$ holds C.F.E

for any $x \in (-\epsilon, \epsilon)$

$\exists r \in \mathbb{Q}$ s.t $x + r \in (a, b)$.

and $g(r) = 0$

$g(x) = g(x) + g(r) = g(x+r) = f(x+y) - (x+y) f(1)$

$= f(t) - t f(1)$

$\Rightarrow$ f is bounded on (a,b). so $(-\epsilon_1, \epsilon)$

$\Rightarrow$ $|f(x)| \le M$,

Let $\langle a_n \rangle \to 0$ and $\langle r_n \rangle$ is a sequence of rational s.t. $\langle r_n \rangle \to \infty$

but $\langle r_n \cdot a_n \rangle \to 0$

$$|f(a_n)| = f\left(\frac{r_n \cdot a_n}{r_n}\right) = \frac{1}{|r_n|} f(r_n \cdot a_n)$$

$$\le \frac{M}{|r_n|} \to 0$$

$\Rightarrow$ $\langle f(a_n) \rangle \to 0$.

$f(x+y) = f(x) \cdot f(y)$, $f: \mathbb{R} \to \mathbb{R}$, $\forall x, y \in \mathbb{R}$.

if f is continuous at one point then f is continuous on $\mathbb{R}$.

<u>**Proof:**</u> if $\exists c \in \mathbb{R}$ s.t. $f(c) = 0$

then for any $x \in \mathbb{R}$.

$f(x) = f(x-c+c) = f(x-c) \cdot f(c) = 0$

$\equiv 0$, $\forall x \in \mathbb{R}$.

<u>case-II</u> $f(x) \neq 0$, for any $x \in \mathbb{R}$.

i.e. $0 \notin f(\mathbb{R})$.

Now

$f(x) = f\left(\frac{x}{2} + \frac{x}{2}\right) = f(x/2) \cdot f(x/2)$

$= (f(x/2))^2 > 0$

$\Rightarrow$ $\forall x \in \mathbb{R} \Rightarrow f(x) > 0$

i.e. $f(\mathbb{R}) \subset (0, \infty)$

$\Rightarrow$ $g(x) = \log_e f(x)$ is defined on $\mathbb{R}$.

$g(x+y) = \log_e f(x+y)$

$= \log_e f(x) \cdot f(y)$

$= \log_e f(x) + \log_e f(y)$

$\Rightarrow$ $g(x)$ holds C.F.E.

$g(x) = \log_e f(x)$.

$f(x) = e^{g(x)}$.

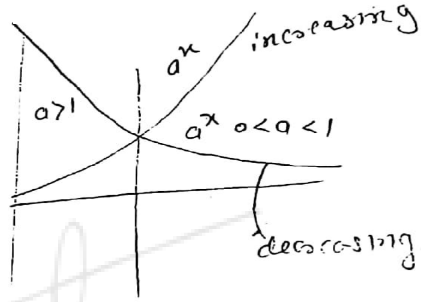
if f is continuous at one. point

$\Rightarrow$ g is continuous at that point

$\Rightarrow$ $g(x) = cx$.

$\Rightarrow$ $f(x) = e^{cx}$.

$\Rightarrow$ $\boxed{f(x) = a^x}$, $(e^c)^x$, $e^c = a$.



\# $f: \mathbb{R} \longrightarrow \mathbb{R}$ such that

$f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2}$ —(1)

Then Eq(I) is called Jensen Equation.

Proof:- $f\left(\frac{x+0}{2}\right) = \frac{f(x)+f(0)}{2}$

$f(x/2) = \frac{f(x)+c}{2}$.

$f(t/2) = \frac{f(t)+c}{2}$, $\forall t \in \mathbb{R}$.

$f\left(\frac{x+y}{2}\right) = \frac{f(x+y)+c}{2}$

$\frac{f(x)+f(y)}{\cancel{2}} = \frac{f(x)+f(y)+c}{2}$

$f(x+y) = f(x) + f(y) + c$

define $g(x) = f(x) - c$

$g(x+y) = f(x+y) - c$

$= f(x) + f(y) - c - c$

$= f(x) - c + f(y) - c$

$= g(x) + g(y)$

$\Rightarrow g(x)$ holds C.F.E

if f is continuous at one point

$\Rightarrow g(x)$ is continuous at that point

then $g(x) = cx$, $\forall x \in \mathbb{R}$

$$\boxed{f(x) = \alpha x + c \quad , \quad \forall x \in \mathbb{R}}$$

Identity Theorem :- if f is continuous on $\mathbb{R}$ and $S \subset \mathbb{R}$ such that $S' = \mathbb{R}$. Then.

if $f(x) = 0$, $\forall x \in S$

$\Rightarrow f(x) \equiv 0$, $\forall x \in \mathbb{R}$.

$\alpha \in \mathbb{R}$

as $S' = \mathbb{R} \Rightarrow \alpha \in S'$

$\exists \langle a_n \rangle \to \alpha$, $a_n \in S - \{\alpha\}$.

$\langle f(a_n) \rangle \to f(\alpha)$

$\to 0 \quad \Rightarrow f(\alpha) = 0$

$\Rightarrow f(x) \equiv 0$, $\forall x \in \mathbb{R}$

Note: f is cont.

$N = Z(f) = \{x \in \mathbb{R} : f(x) = 0\}$.

Let $\alpha \in N'$

$\exists \langle a_n \rangle \to \alpha$. such that $a_n \in N$

$\Rightarrow f(a_n) \to 0$

$\langle f(a_n) \rangle \to f(\alpha)$

$\to 0$

$\Rightarrow f(\alpha) = 0 \Rightarrow \alpha \in N$.

$\Rightarrow N$ is closed i.e. $Z(f)$ is closed. set.

II - Form:

f and g are continuous on $\mathbb{R}$ and $S \in \mathbb{R}$

such $S' = \mathbb{R}$.

if $f(x) = g(x)$, $\forall x \in S$.

$\Rightarrow f(x) = g(x) \quad \forall x \in \mathbb{R}$.

Proof: $h(x) = f(x) - g(x)$.

h is continuous on $\mathbb{R}$

$h(x) = 0, \quad \forall x \in S$.

$h(x) = 0 \quad \forall x \in \mathbb{R}$

$h \equiv 0$

Example: f is $\mathbb{R} \xrightarrow{\text{continuous}} \mathbb{R}$

$f(m + n\sqrt{2}) = 1 \quad \forall m, n \in \mathbb{Z}$

define.

$A = \{x \in \mathbb{R} : f(x) \neq 1\}$

then A:

(i) A is closed but not open

(ii) A is open but not closed.

(iii) A is bounded ✓

(iv) $\sup A \notin \mathbb{R}$. ✓

$S = \{m + n\sqrt{2} : m, n \in \mathbb{Z}\}$

$\Rightarrow S$ is dense in $\mathbb{R}$

$\Rightarrow f$ is continuous

$\Rightarrow f(\mathbb{R}) = 1$

$A = \phi$.

$\Rightarrow A$ is both open and closed

$\sup A = -\infty \notin \mathbb{R}$